

SUPNORM AND f -HÖLDER ESTIMATES FOR $\bar{\partial}$ ON CONVEX DOMAINS OF GENERAL TYPE IN \mathbb{C}^2

TRAN VU KHANH

ABSTRACT. In this paper, we study supnorm and modified Hölder estimates for the integral solution of the $\bar{\partial}$ -equation on a class of convex domains of general type in \mathbb{C}^2 that includes many infinite type examples.

AMS Mathematics Subject Classification (2000): 23A26, 32A30, 32T18, 32T25, 32W05.

Key words and phrases: convex domain; domain of finite/infinite type; f -Hölder, supnorm, L^2 -superlogarithmic estimate.

CONTENTS

1.	Introduction	1
2.	Preliminaries	4
3.	Estimates on $\Omega \cap U = \{\rho(z) = F(z_1 ^2) + r(z) < 0\}$	7
4.	Estimates on $\Omega \cap U = \{\rho(z) = F(\operatorname{Re} z_1 ^2) + r(z) < 0\}$	10
5.	General Hardy-Littlewood Lemma for f -Hölder estimates	12
	Appendix	14
	References	16

1. INTRODUCTION

Let Ω be a smooth, bounded domain in \mathbb{C}^2 with 0 in the boundary $b\Omega$. Assume that Ω is strictly convex except possibly on a neighborhood U of 0; and in U , Ω has the form

$$\Omega \cap U = \{\rho(z) = F(|z_1|^2) + r(z) < 0\} \quad (1.1)$$

or

$$\Omega \cap U = \{\rho(z) = F(|\operatorname{Re} z_1|^2) + r(z) < 0\}. \quad (1.2)$$

where F is a strictly increasing, convex function such that $F(0) = 0$, $F(t)/t$ is increasing, and r is convex with $\frac{\partial r}{\partial z_2} \neq 0$. We remark that Ω may be of finite type or infinite type since we may choose, for example, $F(t) = t^m$ or $F(t) = \exp(-1/t^\alpha)$. The primary goals of this paper are to investigate the supnorm estimate and develop appropriate Hölder

estimate for the integral solution of the $\bar{\partial}$ -equation given by Henkin kernel on a domain Ω satisfying (1.1) or (1.2).

Given a bounded, $\bar{\partial}$ -closed $(0, 1)$ form ϕ , the supnorm and the Hölder estimates for the solution of the Cauchy-Riemann equation

$$\bar{\partial}u = \phi$$

on domain Ω is a fundamental question in several complex variables. A positive answer is well-known when Ω is a

- strongly pseudoconvex domain in \mathbb{C}^n (see [He70], [Ke71], [Ra86] ...),
- convex domain of finite type in \mathbb{C}^n (see [DF06], [DFF99], [H02]...),
- real or complex ellipsoid of finite type in \mathbb{C}^n (see [BC84], [F96], [DFW86],...),
- or a pseudoconvex domain of finite type in \mathbb{C}^2 (see [FK88],[Ra90],[CNS92]...).

However, when Ω is of infinite type, the only result is by J. E. Fornaess, L. Lee, and Y. Zhang [FLZ11] who prove supnorm estimates in the case $F(t) = \exp(-1/t^\alpha)$ with $\alpha < \frac{1}{2}$ and $r(z) = \operatorname{Re} z_2$ for both (1.1) and (1.2). Denote by $A \lesssim B$ for inequality $A \leq cB$ with some positive constant c , for simplification. We denote by $L^\infty(\Omega)$ the space of the essentially bounded functions on Ω and by $\|u\|_\infty$ the essential supremum of $u \in L^\infty(\Omega)$ in Ω .

Theorem 1.1 (Fornaess-Lee-Zhang). *Let Ω be a smooth, bounded domain in \mathbb{C}^2 with 0 in the boundary $b\Omega$. Assume that Ω is strictly convex except possibly on a neighborhood U of 0; and in U , Ω has the form*

$$\Omega \cap U = \{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|z_1|^\alpha) < 0\} \quad (1.3)$$

or

$$\Omega \cap U = \{\rho(z) = \operatorname{Re} z_2 + \exp(-1/|\operatorname{Re} z_1|^\alpha) < 0\}. \quad (1.4)$$

with $\alpha < 1$. Then there is a solution to the $\bar{\partial}$ -equation $\bar{\partial}u = \phi$ for any $\phi \in C_{(0,1)}^1(\bar{\Omega})$ and $\bar{\partial}\phi = 0$, that satisfies $\|u\|_\infty \lesssim \|\phi\|_\infty$.

The first goal of the paper is to prove supnorm estimates on domains satisfying (1.1) or (1.2) which both generalize the class of domains considered in [FLZ11].

Theorem 1.2. (i) *Let Ω and F be as in (1.1). Assume that $\int_0^\delta |\ln F(t^2)| dt < \infty$ for some $\delta > 0$. Then for any bounded $\bar{\partial}$ -closed $(0, 1)$ form ϕ on $\bar{\Omega}$, there is a u such that $\bar{\partial}u = \phi$ on Ω and $\|u\|_\infty \lesssim \|\phi\|_\infty$.*

(ii) Let Ω and F be as in (1.2). Assume that $\int_0^\delta |(\ln t)(\ln F(t^2))|dt < \infty$ for some $\delta > 0$. Then for any bounded $\bar{\partial}$ -closed $(0, 1)$ form ϕ on $\bar{\Omega}$, there is a u such that $\bar{\partial}u = \phi$ on Ω and $\|u\|_\infty \lesssim \|\phi\|_\infty$.

When Ω is finite type (e.g., $F(t) = t^m$), we known that fractional Hölder estimates hold for both case (1.1) and (1.2). However, when Ω is infinite type (e.g., $F(t) = \exp(-1/t^\alpha)$), McNeal [Mc91] proves the fractional Hölder estimates do not hold.

In this paper, we find a suitable Hölder estimate for infinite type. Let f be an increasing function on $(a, +\infty)$ with a big enough such that $\lim_{t \rightarrow +\infty} f(t) = +\infty$. For $\Omega \subset \mathbb{C}^n$, we define f -Hölder space on Ω by

$$\Lambda^f(\Omega) = \{u : \|u\|_\infty + \sup_{z, z+h \in \Omega} f(|h|^{-1}) \cdot |u(z+h) - u(z)| < \infty\}$$

and set

$$\|u\|_f = \|u\|_\infty + \sup_{z, z+h \in \Omega} f(|h|^{-1}) \cdot |u(z+h) - u(z)|.$$

Note that the f -Hölder spaces include the standard Hölder spaces $\Lambda_\alpha(U)$ by taking $f(t) = t^\alpha$ (so $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha < 1$. In this way, f -Hölder spaces generalize the notion of the Hölder spaces.

Since F is strictly increasing F is invertible with inverse F^* . Our main result is

Theorem 1.3. (i) Let Ω and F be defined by (1.1). Assume that $\int_0^\delta |\ln F(t^2)|dt < \infty$ for some $\delta > 0$. Then for every bounded $\bar{\partial}$ -closed $(0, 1)$ form ϕ on $\bar{\Omega}$, there exists a function u in $\Lambda^f(\Omega)$ such that $\bar{\partial}u = \phi$ and

$$\|u\|_f \lesssim \|\phi\|_\infty \quad (1.5)$$

where

$$f(d^{-1}) = \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} dt \right)^{-1}.$$

(ii) Let Ω and F be defined by (1.2). Assume that $\int_0^\delta |(\ln t)(\ln F(t^2))|dt < \infty$ for some $\delta > 0$. Then for every bounded $\bar{\partial}$ -closed $(0, 1)$ form ϕ on $\bar{\Omega}$, there exists a function u in $\Lambda^f(\Omega)$ such that $\bar{\partial}u = \phi$ and

$$\|u\|_f \lesssim \|\phi\|_\infty \quad (1.6)$$

where

$$f(d^{-1}) = \left(\int_0^d \frac{\sqrt{F^*(t)} |\ln \sqrt{F^*(t)}|}{t} dt \right)^{-1}.$$

The following examples are explicit function f in the choice of F .

Example 1.1. Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : F(|z_1|^2) + |z_2 - 1|^2 < 1\}$. Then supnorm and f -Hölder estimates hold for the integral solution of $\bar{\partial}$ -equation in the following examples:

- (1) If $F(t^2) = t^{2m}$, then $f(d^{-1}) = d^{-1/2m}$.
- (2) If $F(t^2) = 2 \exp\left(-\frac{1}{t^\alpha}\right)$ with $0 < \alpha < 1$, then $f(d^{-1}) = (-\ln d)^{\frac{1}{\alpha}-1}$.
- (3) If $F(t^2) = 2 \exp\left(-\frac{1}{t|\ln t|^\alpha}\right)$ with $\alpha > 1$, then $f(d^{-1}) = (\ln(-\ln d))^{\alpha-1}$.

Example 1.2. Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : F(|\operatorname{Re} z_1|^2) + G(|\operatorname{Im} z_1|^2) + |z_2 - 1|^2 < 1\}$ where $G(t) \equiv 0$ in a neighborhood of 0 and there is a positive constant c such that $t \geq c$ if $G(t) \geq 1$. Then supnorm and f -Hölder estimates hold for the integral solution of the $\bar{\partial}$ -equation in the following examples:

- (1) If $F(t^2) = t^{2m}$, then $f(d^{-1}) = d^{-1/2m} |\ln d|^{-1}$.
- (2) If $F(t^2) = \exp\left(-\frac{1}{t^\alpha}\right)$ with $0 < \alpha < 1$, then $f(d^{-1}) = (-\ln d)^{\frac{1}{\alpha}-1} (\ln(-\ln d))^{-1}$.
- (3) If $F(t^2) = \exp\left(-\frac{1}{t|\ln t|^\alpha}\right)$ with $\alpha > 2$, then $f(d^{-1}) = (\ln(-\ln d))^{\alpha-2}$.

Remark 1.4. We remark that superlogarithmic estimates defined by Kohn in [Ko02] for the $\bar{\partial}$ -Laplacian \square or Kohn-Laplacian \square_b , which imply local hypoellipticity of \square and \square_b respectively, hold in both domains defined by (1.1) and (1.2) under hypothesis in Theorem 1.3 (see Appendix).

I am grateful to Andrew Raich for helpful comments on the original draft of this paper.

2. PRELIMINARIES

In this section we briefly recall the construction of an integral kernel to solve the $\bar{\partial}$ -equation on convex domains in \mathbb{C}^2 . Details can be found in [He70], [Ra86].

Let ρ be the defining function of Ω . We can assume that there is a $\delta > 0$ such that $b\Omega \setminus B(0, \delta)$ is strictly convex and

$$\Omega \cap B(0, \delta) = \{\rho(z) = P(z_1) + r(z) < 0\} \quad (2.1)$$

where $P(z_1) = F(|z_1|^2)$ or $F(|\operatorname{Re} z_1|^2)$ and $r(z)$ is convex with $\frac{\partial r}{\partial z_2} \neq 0$ on $\Omega \cap B(0, \delta)$.

Define

$$\Phi(z, \zeta) = 2 \left(\frac{\partial \rho(\zeta)}{\partial \zeta_1} (\zeta_1 - z_1) + \frac{\partial \rho(\zeta)}{\partial \zeta_2} (\zeta_2 - z_2) \right).$$

The following result is a well-known consequence of Taylor's theorem and convexity

Lemma 2.1. *There are suitably small ϵ and c such that*

$$\operatorname{Re} \Phi(z, \zeta) \geq -\rho(z) + \begin{cases} c|z - \zeta|^2 & \zeta \in b\Omega \setminus B(0, \delta) \\ P(z_1) - P(\zeta_1) - 2 \operatorname{Re} \frac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1 - \zeta_1) & \zeta \in b\Omega \cap B(0, \delta) \end{cases} \quad (2.2)$$

for all $z \in \bar{\Omega}$ with $|z - \zeta| \leq \epsilon$.

Choose $\chi \in C^\infty(\mathbb{C}^2 \times \mathbb{C}^2)$ such that $0 \leq \chi \leq 1$, $\chi(z, \zeta) = 1$ for $|z - \zeta| \leq \frac{1}{2}\epsilon$ and $\chi(z, \zeta) = 0$ for $|z - \zeta| \geq \epsilon$. For $j = 1, 2$, define

$$\Phi_j(z, \zeta) = \chi \frac{\partial \rho}{\partial \zeta_j}(\zeta) + (1 - \chi)(\bar{\zeta}_j - \bar{z}_j).$$

Then

$$\Phi^\#(z, \zeta) = \Phi_1(z, \zeta)(\zeta_1 - z_1) + \Phi_2(z, \zeta)(\zeta_2 - z_2)$$

has the following properties for any $\zeta \in b\Omega$:

(i)

$$\operatorname{Re} \Phi^\#(z, \zeta) \geq -\rho(z) + \begin{cases} c|z - \zeta|^2 & \zeta \in b\Omega \setminus B(0, \delta), \\ P(z_1) - P(\zeta_1) - 2 \operatorname{Re} \frac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1 - \zeta_1) & \zeta \in b\Omega \cap B(0, \delta); \end{cases} \quad (2.3)$$

for $|z - \zeta| \leq \frac{1}{2}\epsilon$ and $z \in \bar{\Omega}$.

(ii) $\Phi^\#(\cdot, \zeta)$ and $\Phi_j(\cdot, \zeta)$, $j = 1, 2$, are holomorphic on $\{z : |z - \zeta| \leq \frac{1}{2}\epsilon\}$.

We now are ready for the integral solution of the $\bar{\partial}$ -equation. Let $\phi = \phi_1 d\bar{z}_1 + \phi_2 d\bar{z}_2$ be a bounded $\bar{\partial}$ -closed $(0, 1)$ -form on $\bar{\Omega}$. The Hekim integral solution u of the $\bar{\partial}$ -equation $\bar{\partial}u = \phi$ given by

$$u = T\phi(z) = H\phi(z) + K\phi(z).$$

where

$$\begin{aligned} H\phi(z) &= -\frac{1}{2\pi i} \int_{\zeta \in b\Omega} \frac{\Phi_1(z, \zeta)(\bar{\zeta}_2 - \bar{z}_2) - \Phi_2(z, \zeta)(\bar{\zeta}_1 - \bar{z}_1)}{\Phi^\#(z, \zeta)|\zeta - z|^2} \phi \wedge \omega(\zeta); \\ K\phi(z) &= -\frac{1}{2\pi i} \int_{\Omega} \frac{\phi_1(\bar{\zeta}_2 - \bar{z}_2) - \phi_2(\bar{\zeta}_1 - \bar{z}_1)}{|\zeta - z|^4} \omega(\bar{\zeta}) \wedge \omega(\zeta) \end{aligned} \quad (2.4)$$

where $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$.

It is well known that

$$\|K\phi\|_\infty \lesssim \|\phi\|_\infty \quad \text{and} \quad \|K\phi\|_f \lesssim \|\phi\|_\infty,$$

for any $0 < f(d^{-1}) < d^{-1}$ with any $d > 0$ small enough (see Lemma 1.15, page 157 in [Ra86]). Moreover, we have

$$H\phi(z) = -\frac{1}{2\pi i} \int_{\zeta \in b\Omega} \cdots = -\frac{1}{2\pi i} \left(\int_{\zeta \in b\Omega, |z-\zeta| \leq \epsilon} \cdots + \int_{\zeta \in b\Omega, |z-\zeta| \geq \epsilon} \cdots \right) = -\frac{1}{2\pi i} \int_{\zeta \in b\Omega, |z-\zeta| \leq \epsilon} \cdots$$

Since $\Phi_1(z, \zeta)(\bar{\zeta}_2 - \bar{z}_2) - \Phi_2(z, \zeta)(\bar{\zeta}_1 - \bar{z}_1) \equiv 0$ if $|z - \zeta| \leq \epsilon$.

Therefore, it is sufficient to estimate

$$H\phi(z) = -\frac{1}{2\pi i} \int_{\zeta \in b\Omega, |z-\zeta| \leq \epsilon} \frac{\Phi_1(z, \zeta)(\bar{\zeta}_2 - \bar{z}_2) - \Phi_2(z, \zeta)(\bar{\zeta}_1 - \bar{z}_1)}{\Phi(z, \zeta)|\zeta - z|^2} \phi \wedge \omega(\zeta).$$

We will use the general Hardy-Littlewood lemma (see Section 5 below) to obtain the f -Hölder estimates. To do that we need to control the gradient of $T\phi(z)$. We have

$$|\nabla H\phi(z)| \lesssim \|\phi\|_\infty \int_{\zeta \in b\Omega, |z-\zeta| \leq \epsilon} \left(\frac{1}{|\Phi| \cdot |\zeta - z|^2} + \frac{1}{|\Phi|^2 \cdot |\zeta - z|} \right) dS$$

where dS is surface area measure on $b\Omega$. We now use Lemma 2.1 to obtain

$$\int_{\zeta \in b\Omega \setminus B(0, \delta), |z-\zeta| \leq \epsilon} \left(\frac{1}{|\Phi| \cdot |\zeta - z|^2} + \frac{1}{|\Phi|^2 \cdot |\zeta - z|} \right) dS \lesssim \delta^{-1/2}(z).$$

Hence, it remains to estimate

$$L(z) = \int_{\zeta \in b\Omega \cap B(0, \delta), |z-\zeta| \leq \epsilon} \left(\frac{1}{|\Phi| \cdot |\zeta - z|^2} + \frac{1}{|\Phi|^2 \cdot |\zeta - z|} \right) dS.$$

Set $t = \text{Im } \Phi(z, \zeta)$. It is easy to check that $\frac{\partial t}{\partial \zeta_2} \neq 0$. So we change coordinate and obtain

$$\begin{aligned} L(z) &\lesssim \int_{|t| \leq \delta, |\zeta_1| < \delta, |z_1 - \zeta_1| \leq \epsilon} \frac{dtd(\text{Re } \zeta_1)d(\text{Im } \zeta_1)}{(|t| + |\text{Re } \Phi|)(|\rho(z)|^2 + |\zeta_1 - z_1|^2)} \\ &\quad + \int_{|t| \leq \delta, |\zeta_1| < \delta, |z_1 - \zeta_1| \leq \epsilon} \frac{dtd(\text{Re } \zeta_1)d(\text{Im } \zeta_1)}{(t^2 + |\text{Re } \Phi|^2)(|\rho(z)| + |\zeta_1 - z_1|)} \\ &\lesssim |\ln(|\text{Re } \Phi|) \cdot \ln \rho(z)| + \int_{|\zeta_1| < \delta, |z_1 - \zeta_1| \leq \epsilon} \frac{d(\text{Re } \zeta_1)d(\text{Im } \zeta_1)}{|\text{Re } \Phi| \cdot |\zeta_1 - z_1|} \\ &\lesssim |\ln \rho(z)|^2 + \int_{|\zeta_1| < \delta, |z_1 - \zeta_1| \leq \epsilon} \frac{d(\text{Re } \zeta_1)d(\text{Im } \zeta_1)}{|\text{Re } \Phi| \cdot |\zeta_1 - z_1|}. \end{aligned} \tag{2.5}$$

Here the last inequality follows by $|\text{Re } \Phi| \geq |\rho(z)|$ for all $\zeta \in b\Omega \cap B(0, \delta)$ and $|z - \zeta| \leq \epsilon$ which is itself a consequence of Lemma 2.1 and the convexity of P (see (3.2) and (4.1) below).

We have therefore show

$$|\nabla H\phi(z)| \lesssim \left(\rho(z)^{-1/2} + \int_{|\zeta_1| < \delta, |\zeta_1 - z_1| < \epsilon} \frac{d(\operatorname{Re} \zeta_1) d(\operatorname{Im} \zeta_1)}{|\operatorname{Re} \Phi| \cdot |\zeta_1 - z_1|} \right) \|\phi\|_\infty. \quad (2.6)$$

A similar argument also shows

$$|H\phi(z)| \lesssim \left(1 + \int_{|\zeta_1| < \delta, |\zeta_1 - z_1| < \epsilon} \frac{|\ln |\operatorname{Re} \Phi|| d(\operatorname{Re} \zeta_1) d(\operatorname{Im} \zeta_1)}{|\zeta_1 - z_1|} \right) \|\phi\|_\infty. \quad (2.7)$$

3. ESTIMATES ON $\Omega \cap U = \{\rho(z) = F(|z_1|^2) + r(z) < 0\}$

In this section, we give the proof of Theorem 1.2.(i) and Theorem 1.3.(ii). It is sufficient to estimate the integrals in (2.6) and (2.7) when $z \in B(0, \delta)$ so the defining function ρ is of the form $\rho(z) = F(|z_1|^2) + r(z)$ in $B(0, \delta)$.

Lemma 3.1. *Let F be a convex, C^2 -smooth function on $[0, \delta]$. Then we have*

$$F(p) - F(q) - F'(q)(p - q) \geq 0$$

for any $p, q \in [0, \delta]$.

The proof is simple and is omitted here.

Lemma 3.2. *For $\delta > 0$ small enough, let F be an invertible on $[0, \delta]$ such that $\frac{F(t)}{t}$ is increasing $[0, \delta]$. Then*

$$\int_0^\delta \frac{dr}{\varrho + F(r^2)} \lesssim \frac{\sqrt{F^*(\varrho)}}{\varrho}$$

for any sufficiently small $\varrho > 0$.

Proof. We split our integration to be two terms

$$\int_0^\delta \frac{dr}{\varrho + F(r^2)} = \int_0^{\sqrt{F^*(\varrho)}} \dots + \int_{\sqrt{F^*(\varrho)}}^\delta \dots$$

For the first term, it is easy to see that

$$\int_0^{\sqrt{F^*(\varrho)}} \frac{dr}{\varrho + F(r^2)} \leq \frac{\sqrt{F^*(\varrho)}}{\varrho}.$$

Since $\frac{F(t)}{t}$ is increasing, we have

$$\frac{F(r^2)}{r^2} \geq \frac{F(F^*(\varrho))}{F^*(\varrho)} = \frac{\varrho}{F^*(\varrho)}, \quad \text{or} \quad \frac{F(r^2)}{\varrho} \geq \frac{r^2}{F^*(\varrho)},$$

for any $r \geq \sqrt{F^*(\varrho)}$. Apply this inequality to the second term, we obtain

$$\begin{aligned} \int_{\sqrt{F^*(\varrho)}}^{\delta} \frac{dr}{\varrho + F(r^2)} &\leq \frac{1}{\varrho} \int_{\sqrt{F^*(\varrho)}}^{\delta} \frac{dr}{1 + r^2/F^*(\varrho)} \\ &\leq \frac{\sqrt{F^*(\varrho)}}{\varrho} \int_1^{\infty} \frac{dy}{1 + y^2} = \frac{\pi}{4} \frac{\sqrt{F^*(\varrho)}}{\varrho} \end{aligned} \quad (3.1)$$

This is complete the proof of Lemma 3.2. \square

Proof of Theorem 1.2.(i). We omit the proof of Theorem 1.2.(i) since it follows in exactly method of the proof of Theorem 1.3.(i) with simpler calculation. \square

Proof of Theorem 1.3.(i). We apply the identity $2 \operatorname{Re} a\bar{b} = |a + b|^2 - |a|^2 - |b|^2$ in (2.2) to obtain \square

$$\begin{aligned} \operatorname{Re} \Phi(z, \zeta) &\geq -\rho(z) + F(|z_1|^2) - F(|\zeta_1|^2) + 2F'(|\zeta_1|^2) \operatorname{Re} (\bar{\zeta}_1(z_1 - \zeta_1)) \\ &\geq -\rho(z) + \left(F'(|\zeta_1|^2)|z_1 - \zeta_1|^2 + F(|z_1|^2) - F(|\zeta_1|^2) - F'(|\zeta_1|^2)(|z_1|^2 - |\zeta_1|^2) \right) \\ &\geq -\rho(z) + \left(F'(|\zeta_1|^2)|z_1 - \zeta_1|^2 \right) \end{aligned} \quad (3.2)$$

where the last inequality follows by Lemma 3.1.

Let $M(z)$ be the integral term in (2.6). We will show that

$$M(z) \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|} \quad (3.3)$$

for $z \in \Omega$. For convenient, set $\varrho = |\rho(z)| > 0$ when $z \in \Omega$. From (3.2), we have

$$M(z) \leq \int_{|\zeta_1| < \delta, |\zeta_1 - z_1| < \epsilon} \frac{d(\operatorname{Re} \zeta_1) d(\operatorname{Im} \zeta_1)}{(\varrho + F'(|\zeta_1|^2)|z_1 - \zeta_1|^2)|\zeta_1 - z_1|}.$$

There are now two cases.

Case 1: $|z_1 - \zeta_1| \geq |\zeta_1|$. In this case,

$$(\varrho + F'(|\zeta_1|^2)|z_1 - \zeta_1|^2)|z_1 - \zeta_1| \geq (\varrho + F'(|\zeta_1|^2)|\zeta_1|^2)|\zeta_1| \geq (\varrho + F'(|\zeta_1|^2))|\zeta_1|.$$

Here the last inequality follows from the inequality $tF'(t) \geq F(t)$ which is itself a consequence of the fact that $\frac{F(t)}{t}$ is increasing. Therefore, using polar coordinates and Lemma

3.2,

$$\begin{aligned}
M(z) &\leq \int_{|\zeta_1| < \delta} \frac{d(\operatorname{Re} \zeta_1) d(\operatorname{Im} \zeta_1)}{(\varrho + F(|\zeta_1|^2)) |\zeta_1|} \\
&\lesssim \int_0^\delta \frac{dr}{\varrho + F(r^2)} \\
&\lesssim \frac{\sqrt{F^*(\varrho)}}{\varrho}.
\end{aligned} \tag{3.4}$$

Case 2: If $|\zeta_1| \geq |z_1 - \zeta_1|$, then the fact that F' is increasing (F is convex) implies

$$F'(|\zeta_1|^2) |z_1 - \zeta_1|^2 \geq F'(|z_1 - \zeta_1|^2) |z_1 - \zeta_1|^2 \gtrsim F(|z_1 - \zeta_1|^2).$$

Similarly, we obtain

$$\begin{aligned}
M(z) &\leq \int_{|\zeta_1 - z_1| < \epsilon} \frac{d(\operatorname{Re} \zeta_1) d(\operatorname{Im} \zeta_1)}{(\varrho + F(|\zeta_1 - z_1|^2)) |\zeta_1 - z_1|} \\
&\lesssim \int_0^\epsilon \frac{dr}{\varrho + F(r^2)} \\
&\lesssim \frac{\sqrt{F^*(\varrho)}}{\varrho}
\end{aligned} \tag{3.5}$$

The proof of (3.3) is complete. Combining (2.6) and (3.3), we obtain

$$|\nabla T(\phi)| \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|} \|\phi\|_\infty \lesssim \frac{\sqrt{F^*(\delta_{b\Omega}(z))}}{\delta_{b\Omega}(z)} \|\phi\|_\infty. \tag{3.6}$$

since the distance $\delta_{b\Omega}(z)$ is comparable to $|\rho(z)|$.

Finally, to apply the general Hardy-Littlewood Lemma (see Section 5), we need to check that $G(t) := \sqrt{F^*(t)}$ satisfies the hypothesis of Theorem 5.1. It is easy to see that $\sqrt{F^*(t)}$ is increasing and $\frac{\sqrt{F^*(t)}}{t}$ is decreasing. For δ small enough, $|\ln(F(t^2))|$ is decreasing when $0 \leq t \leq \delta$ so we can estimate

$$|\ln F(\eta^2)| \eta \leq \int_0^\eta |\ln F(t^2)| dt \leq \int_0^\delta |\ln F(t^2)| dt < \infty$$

for any $0 \leq \eta \leq \delta$. The integral is finite by the hypothesis. Consequently, $\sqrt{F^*(t)}|\ln t| < \infty$ for any $0 \leq t \leq \sqrt{F^*(\delta)}$ and $\lim_{t \rightarrow 0} t|\ln F(t^2)| = 0$. This implies

$$\begin{aligned} \int_0^d \frac{\sqrt{F^*(t)}}{t} dt &\stackrel{y:=\sqrt{F^*(t)}}{=} \int_0^{\sqrt{F^*(d)}} y (\ln F(y^2))' dy \\ &= \sqrt{F^*(d)} \ln d - \int_0^{\sqrt{F^*(d)}} (\ln F(y^2)) dy < \infty. \end{aligned} \quad (3.7)$$

for d sufficiently small. Here, the integral in (3.7) is finite by the hypothesis. Thus the proof of Theorem 1.3.(i) is complete. \square

4. ESTIMATES ON $\Omega \cap U = \{\rho(z) = F(|\operatorname{Re} z_1|^2) + r(z) < 0\}$

In this section, we give the proof of Theorem 1.2.(ii) and Theorem 1.3.(ii). It is sufficient to estimate the integrations in (2.6) and (2.7) when the defining function of Ω in a neighborhood of 0 has the form $\rho = F(|\operatorname{Re} z_1|^2) + r(z)$.

We set $x_1 = \operatorname{Re} z_1$, $y_1 = \operatorname{Im} z_1$, $\xi_1 = \operatorname{Re} \zeta_1$ and $\eta_1 = \operatorname{Im} \zeta_1$. From (2.2), we have

$$\begin{aligned} \operatorname{Re} \Phi(z, \zeta) &\geq -\rho(z) + \left(F(x_1^2) - F(\xi_1^2) - 2F'(\xi_1^2)\xi_1(x_1 - \xi_1) \right) \\ &\geq -\rho(z) + F'(\xi_1^2)(x_1 - \xi_1)^2 + \left(F(x_1^2) - F(\xi_1^2) - F'(\xi_1^2)(x_1^2 - \xi_1^2) \right) \\ &\geq -\rho(z) + F'(\xi_1^2)(x_1 - \xi_1)^2 \end{aligned} \quad (4.1)$$

where the last inequality follows by Lemma 3.1.

Proof of Theorem 1.2.(ii). We only need to show that the integral term in (2.7) is bounded. By the estimates of $\operatorname{Re} \Phi(z, \zeta)$ as above, we get

$$\begin{aligned}
& \int_{|\zeta_1| < \delta, |\zeta_1 - z_1| < \epsilon} \frac{|\ln |\operatorname{Re} \Phi|| d(\operatorname{Re} \zeta_1) d(\operatorname{Im} \zeta_1)}{|\zeta_1 - z_1|} \\
& \lesssim \int_{|\zeta_1| < \delta, |\zeta_1 - z_1| < \epsilon} \frac{|\ln(F'(\xi_1^2)(x_1 - \xi_1)^2)| d\xi_1 d\eta_1}{|x_1 - \xi_1| + |y_1 - \eta_1|} \\
& \lesssim \int_{|x_1| < \delta, |x_1 - \xi_1| < \epsilon} |\ln |x_1 - \xi_1| \cdot \ln(F'(\xi_1^2)(x_1 - \xi_1)^2)| dx_1 \\
& \lesssim \int_{|x_1| < \delta, |x_1 - \xi_1| < \epsilon; |\xi_1| \leq |x_1 - \xi_1|} \dots + \int_{|x_1| < \delta, |x_1 - \xi_1| < \epsilon; |\xi_1| \geq |x_1 - \xi_1|} \dots \quad (4.2) \\
& \lesssim \int_{|x_1| < \delta} |\ln |\xi_1| \cdot \ln(F'(\xi_1^2)\xi_1^2)| dx_1 \\
& \quad + \int_{|x_1 - \xi_1| < \epsilon} |\ln |x_1 - \xi_1| \cdot \ln(F'((x_1 - \xi_1)^2)(x_1 - \xi_1)^2)| dx_1 \\
& \lesssim \int_{|t| < \max\{\delta, \epsilon\}} |\ln |t| \cdot \ln(F(t^2))| dt < \infty
\end{aligned}$$

Here, the first inequality in the last line of (4.2) follows from the inequality $t^2 F(t^2) \geq F(t^2)$ and the last one of this line follows by hypothesis of theorem. This completes the proof of Theorem 1.2.(ii). \square

Proof of Theorem 1.3.(ii). We only need to estimate of the integral term in (2.6). By the estimates of $\operatorname{Re} \Phi(z, \zeta)$ above, we observe

$$\begin{aligned}
& \int_{|\zeta_1| < \delta, |\zeta_1 - z_1| < \epsilon} \frac{d\xi_1 d\eta_1}{(\varrho + F'(\xi_1^2)(x_1 - \xi_1)^2)(|x_1 - \xi_1| + |y_1 - \eta_1|)} \\
& \lesssim \int_{|\xi_1| < \delta, |x_1 - \xi_1| < \epsilon} \frac{|\ln |x_1 - \xi_1|| d\xi_1}{\varrho + F'(\xi_1^2)(x_1 - \xi_1)^2} \quad (4.3) \\
& \lesssim \int_{|t| < \max\{\delta, \epsilon\}} \frac{|\ln t| dt}{\varrho + F(t^2)}
\end{aligned}$$

Here, the last inequality in the last line of (4.3) follows by the comparison of $|\xi_1|$ and $|x_1 - \xi_1|$; and the property $t^2 F'(t^2) \geq F(t^2)$ as in Theorem 1.2.(ii). To estimate the integral term in the last line of (4.3) we need following lemma.

Lemma 4.1. *For $\delta > 0$ small enough, let F be an invertible on $[0, \delta]$ such that $\frac{F(t)}{t}$ is increasing $[0, \delta]$. Then*

$$\int_0^\delta \frac{|\ln t| dt}{\varrho + F(t^2)} \lesssim \frac{\sqrt{F^*(\varrho)} |\ln \sqrt{F^*(\varrho)}|}{\varrho}$$

for any $\varrho > 0$ sufficiently small.

Proof of Lemma 4.1. Proof. We split our integration into two terms

$$\int_0^\delta \frac{|\ln t| dt}{\varrho + F(t^2)} = \int_0^{\sqrt{F^*(\varrho)}} \dots + \int_{\sqrt{F^*(\varrho)}}^\delta \dots \quad (4.4)$$

For the first term, we have

$$\int_0^{\sqrt{F^*(\varrho)}} \dots \leq \frac{1}{\varrho} \int_0^{\sqrt{F^*(\varrho)}} |\ln t| dt \lesssim \frac{\sqrt{F^*(\varrho)} |\ln \sqrt{F^*(\varrho)}|}{\varrho}.$$

For the second term

$$\int_{\sqrt{F^*(\varrho)}}^\delta \dots \leq |\ln \sqrt{F^*(\varrho)}| \int_{\sqrt{F^*(\varrho)}}^\delta \frac{dt}{\varrho + F(t^2)} \lesssim \frac{\sqrt{F^*(\varrho)} |\ln \sqrt{F^*(\varrho)}|}{\varrho}$$

where the last inequality follows by (3.1). This is the proof of Lemma 4.1. \square

Similarly to the proof of (3.7) we obtain $\int_0^d \frac{\sqrt{F^*(t)} |\ln \sqrt{F^*(t)}|}{t} dt < \infty$ under hypothesis $\int_0^\delta |\ln t \cdot \ln F(t^2)| dt < \infty$ for $d, \delta > 0$ enough small.

Using the general Hardy-Littlewood Lemma, we obtain the proof of Theorem 1.3.(ii). \square

5. GENERAL HARDY-LITTLEWOOD LEMMA FOR f -HÖLDER ESTIMATES

We conclude by proving a general Hardy-Littlewood Lemma for f -Hölder estimates.

Theorem 5.1. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^N and let $\delta_{b\Omega}(x)$ denote the distance function from x to the boundary of Ω . Let $G : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a increasing function such that $\frac{G(t)}{t}$ is decreasing and $\int_0^d \frac{G(t)}{t} dt < \infty$ for $d > 0$ small enough. If $u \in C^1(\Omega)$ such that*

$$|\nabla u(x)| \lesssim \frac{G(\delta_{b\Omega}(x))}{\delta_{b\Omega}(x)} \quad \text{for every } x \in \Omega. \quad (5.1)$$

Then $|u(x) - u(y)| \lesssim f(|x - y|^{-1})^{-1}$, for $x, y \in \Omega, x \neq y$ where $f(d^{-1}) = \left(\int_0^d \frac{G(t)}{t} dt \right)^{-1}$.

Remark 5.2. If $G(t) = t^\alpha$, Theorem 5.1 is the usual Hardy-Littlewood Lemma for domains of finite type. The proof of this theorem in this case can be found in [CS01].

Proof. Since $u \in C^1$ in the interior of Ω , we only need to prove the assertion when z and w are near the boundary. Using a partition of unity, we can assume that u is supported in $U \cap \bar{\Omega}$, where U is a neighborhood of a boundary point $x_o \in b\Omega$. After linear change of coordinates, we may assume $x_o = 0$ and for some $\delta > 0$,

$$U \cap \Omega = \{x = (x', x_N) | x_N > \phi(x'), |x'| < \delta, |x_N| \leq \delta\},$$

where $\phi(0) = 0$ and ϕ is some Lipschitz function with Lipschitz constant M . Let $x = (x, x_N), y = (y', y_N) \in \Omega$ and $d = |x - y|$. For $a \geq 0$, we define the line segment L_a by $\theta(x', x_N + a) + (1 - \theta)(y', y_N + a)$, $0 \leq \theta \leq 1$. Using the Lipschitz property of ϕ , we obtain

$$\begin{aligned} \tilde{x}_N + Md &= \theta(x_N + Md) + (1 - \theta)(y_N + Md) \\ &\geq Md + \theta\phi(x') + (1 - \theta)\phi(y') \\ &\geq Md + \theta(\phi(x') - \phi(\tilde{x}')) + (1 - \theta)(\phi(y') - \phi(\tilde{x}')) + \phi(\tilde{x}') \\ &\geq \phi(\tilde{x}'). \end{aligned} \tag{5.2}$$

This implies that L_a lies in Ω for any $a \geq Md$. Since $u \in C^1(\Omega)$, the Mean Value Theorem tells us that there must exist some $(\tilde{x}', \tilde{x}_N + 2Md) \in L_{2Md}$ such that

$$|u(x', x_N + 2Md) - u(y', y_N + 2Md)| \leq |\nabla u(\tilde{x}', \tilde{x}_N + 2Md)|d.$$

The distance function $\delta_{b\Omega}(x', x_N)$ is comparable to $x_N - \phi(x')$, i.e., there are positive constants c, C such that

$$c(x_N - \phi(x')) \leq \delta_{b\Omega}(x', x_N) \leq C(x_N - \phi(x')) \text{ for } x \in \Omega \cap U. \tag{5.3}$$

Using hypothesis of G , combining with (5.1) and (5.3), it follows that

$$\begin{aligned} |u(x', x_N + 2Md) - u(y', y_N + 2Md)| &\lesssim \frac{G(\delta_{b\Omega}(\tilde{x}', \tilde{x}_N + 2Md))}{\delta_{b\Omega}(\tilde{x}', \tilde{x}_N + 2Md)} d \\ &\lesssim \frac{G(c(\tilde{x}_N + 2Md - \phi(\tilde{x}')))}{c(\tilde{x}_N + 2Md - \phi(\tilde{x}'))} d \\ &\lesssim \frac{G(cMd)}{cMd} d \\ &\lesssim G(d), \end{aligned} \tag{5.4}$$

where the last inequality follows by considering two case of cM ; if $cM < 1$, we use $G(t)$ increasing; otherwise, we use $\frac{G(t)}{t}$ decreasing. We also have

$$\begin{aligned} |u(x) - u(x', x_N + 2Md)| &= \left| \int_0^d \frac{\partial u(x', x_N + 2Mt)}{\partial t} dt \right| \\ &\lesssim \int_0^d \frac{G(\delta_{b\Omega}(x', x_N + 2Mt))}{\delta_{b\Omega}(x', x_N + 2Mt)} dt \\ &\lesssim \int_0^d \frac{G(t)}{t} dt. \end{aligned} \quad (5.5)$$

Thus for any $x, y \in \Omega$,

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x', x_N + 2Md)| + |u(y) - u(y', y_N + 2Md)| \\ &\quad + |u(x', x_N + 2Md) - u(y', y_N + 2Md)| \\ &\lesssim G(d) + \int_0^d \frac{G(t)}{t} dt \lesssim \int_0^d \frac{G(t)}{t} dt. \end{aligned} \quad (5.6)$$

Here, the last inequality follows from

$$G(d) = \int_0^d \frac{G(d)}{d} dt \leq \int_0^d \frac{G(t)}{t} dt.$$

This proves the theorem. □

APPENDIX

In this part, we give an explanation of Remark 1.4. First we show the following theorem.

Theorem 5.3. *Let Ω and F be defined by (1.1) or (1.2) and let $f(d^{-1}) = (\sqrt{F^*(d)})^{-1}$ (for $d > 0$ small enough). Then f -estimate holds for the $\bar{\partial}$ -Neumann problem, that is,*

$$\|f(\Lambda)u\|^2 \lesssim \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2, \quad (5.7)$$

for any $u \in C_{(0,1)}^\infty(\bar{\Omega}) \cap \text{Dom}(\bar{\partial}^*)$, where $\|\cdot\|$ is the $L^2(\Omega)$ norm, $f(\Lambda)$ is the tangential pseudo-differential operator with symbol $f((1 + |\xi|^2)^{1/2})$ and $\bar{\partial}^*$ is the L^2 -adjoint of $\bar{\partial}$ with its domain $\text{Dom}(\bar{\partial}^*)$.

Proof. We will only give the proof in the case Ω is defined by (1.2), that is,

$$\Omega \cap U = \{\rho(z) = F(x_1^2) + r(z) < 0\},$$

since the other is proved similarly. Here, $x_1 = \operatorname{Re} z_1$. It is sufficient to show that there exists a family of absolutely bounded weights $\{\Phi^\delta\}$ defined on $S_\delta \cap U$ satisfying

$$\sum_{i,j=1}^2 \frac{\partial^2 \Phi^\delta}{\partial z_i \partial \bar{z}_j} u_i \bar{u}_j \gtrsim f(\delta^{-1})^2 |u|^2 \quad \text{on } S_\delta \cap U \quad (5.8)$$

for any $u \in C_{(0,1)}^\infty(\bar{\Omega} \cap U)$, where $S_\delta = \{z \in \Omega : -\delta \leq \rho(z) \leq 0\}$ and U is a neighborhood of the origin (see Theorem 1.4 in [KZ10]).

For any $\delta > 0$, we define

$$\Phi^\delta(z) := \exp\left(\frac{\rho(z)}{\delta} + 1\right) - \exp\left(-\frac{x_1^2}{4F^*(\delta)}\right).$$

The weights Φ^δ are absolutely bounded on $S_\delta \cap U$. Computing of the Levi form of Φ^δ shows that

$$\begin{aligned} \sum_{i,j=1}^2 \frac{\partial^2 \Phi^\delta(z)}{\partial z_i \partial \bar{z}_j} u_i \bar{u}_j &= \frac{1}{\delta} \left(\sum_{i,j=1}^2 \frac{\partial^2 \rho(z)}{\partial z_i \partial \bar{z}_j} u_i \bar{u}_j + \frac{1}{\delta} \left| \sum_{j=1}^2 \frac{\partial \rho(z)}{\partial z_j} u_j \right|^2 \right) \exp\left(\frac{\rho(z)}{\delta} + 1\right) \\ &\quad + \frac{1}{8F^*(\delta)} \left(1 - \frac{x_1^2}{2F^*(\delta)}\right) \exp\left(-\frac{x_1^2}{4F^*(\delta)}\right) |u_1|^2 \\ &\geq \frac{1}{2} \left[\frac{1}{\delta} \frac{F(x_1^2)}{x_1^2} + \frac{1}{4F^*(\delta)} \left(1 - \frac{x_1^2}{2F^*(\delta)}\right) \exp\left(-\frac{x_1^2}{4F^*(\delta)}\right) - \frac{c}{F^*(\delta)} \right] |u_1|^2 \\ &\quad + cF^*(\delta)^{-1} |u_2|^2, \end{aligned} \quad (5.9)$$

for any $z \in S_\delta \cap U$, where $c > 0$ will be chosen small. Here, the inequality follows by the hypothesis of ρ and F .

We use the notation that

$$A = \frac{1}{\delta} \frac{F(x_1^2)}{x_1^2}, \quad B = \frac{1}{4F^*(\delta)} \left(1 - \frac{x_1^2}{2F^*(\delta)}\right) \exp\left(-\frac{x_1^2}{4F^*(\delta)}\right), \quad \text{and } C = -\frac{c}{F^*(\delta)}.$$

We consider two cases.

Case 1. $x_1^2 \leq F^*(\delta)$. We have $B \geq \frac{e^{-1/4}}{8F^*(\delta)}$, and hence $A + B + C \gtrsim (F^*(\delta))^{-1}$ for a small choice of c in term C .

Cases 2. Otherwise, assume $x_1^2 \geq F^*(\delta)$. Using our assumption that $\frac{F(t)}{t}$ is increasing, we get

$$A = \frac{1}{\delta} \frac{F(x_1^2)}{x_1^2} \geq \frac{1}{\delta} \frac{F(F^*(\delta))}{F^*(\delta)} = \frac{1}{\delta} \frac{\delta}{F^*(\delta)} = \frac{1}{F^*(\delta)}$$

In this case, B can be negative; however, by using the fact that $\min_{t \geq 1/2} \{(1-t)e^{-t/2}\} = -2e^{-3/2}$ for $t = \frac{x_1^2}{2F^*(\delta)} \geq \frac{1}{2}$, we have $B \geq -\frac{e^{-3/2}}{2F^*(\delta)}$. This implies $A + B + C \gtrsim (F^*(\delta))^{-1}$ for c small enough

Therefore, we obtain (5.8). That concludes the proof of Theorem 5.3. \square

By the equivalence of an f -estimate on a domain and its boundary in \mathbb{C}^2 (see [Kh10]), we have

$$\|f(\Lambda)u\|^2 \lesssim \|\bar{\partial}_b u\|^2 + \|\bar{\partial}_b^* u\|^2 \quad (5.10)$$

holds for any $u \in C_{(0,0)}^\infty(b\Omega)$ or $u \in C_{(0,1)}^\infty(b\Omega)$. Here, the norm in (5.10) is L^2 -norm on $b\Omega$ and $\bar{\partial}_b$ is tangential Cauchy-Riemann operator on $b\Omega$ with its adjoint $\bar{\partial}_b^*$.

Next, we notice that the hypothesis in Theorem 1.3 implies $\lim_{t \rightarrow 0} (t \ln F(t^2)) = 0$. This limit is equivalent to $\lim_{\delta \rightarrow 0} \frac{f(\delta^{-1})}{\log \delta} = \infty$. This means superlogarithmic estimates (in the sense of Kohn [Ko02]) for \square and \square_b hold. Until now, we did not know if there is a function f such that f -Hölder estimate for the integral solution of $\bar{\partial}$ -equation on Ω (defined by (1.1) or (1.2)) holds when $F(t^2) = \exp\left(-\frac{1}{t|\ln t|^\alpha}\right)$ with $0 < \alpha \leq 1$. However, in this case, L^2 -superlogarithmic estimates for \square and \square_b still hold.

REFERENCES

- [BC84] **J. Bruna, J. Castillo**—Hölder and L^p estimates for the $\bar{\partial}$ equation in some convex domains with real analytic boundary, *Math. Ann.*, **269** (1984), 527–539.
- [CNS92] **D. C. Chang, A. Nagel, E. M. Stein**—Estimates for the $\bar{\partial}$ -Neumann problem in pseudoconvex domains of finite type in \mathbb{C}^2 , *Acta Math.*, **169**(1992), 153–228.
- [CS01] **S. C. Chen, M. C. Shaw**—Partial differential equations in several complex variables, *AMS/IP Studies in Advanced Mathematics*, 19. American Mathematical Society, Providence, RI; International Press, Boston, MA, (2001), xii+380 pp.
- [DF06] **K. Diederich, B. Fischer**—Hölder estimates on lineally convex domains of finite type, *Michigan Math. J.*, **54** (2006), 341–352.
- [DFF99] **K. Diederich, B. Fischer, J. E. Fornaess**—Hölder estimates on convex domains of finite type, *Math. Z.*, **232** (1999), 43–61.
- [DFW86] **K. Diederich, J. E. Fornaess, J. Wiegnerinck**—Sharp Hölder estimates for $\bar{\partial}$ on ellipsoids, *Manuscripta Math.*, **56** (1986), 399–413.
- [F96] **J. F. Fleron**—Sharp Hölder estimates for $\bar{\partial}$ on ellipsoids and their complements via order of contact, *Proc. Amer. Math. Soc.*, **124** (1996), 3193–3202.
- [FK88] **C. L. Fefferman, J. J. Kohn**—Hölder estimates on domains of complex dimension two and on three dimensional CR manifolds, *Adv. Math.*, **62** (1988), 223–303.

- [FLZ11] **J. E. Fornaess, L. Lee, Y. Zhang**—On supnorm estimates for $\bar{\partial}$ on infinite type convex domains in \mathbb{C}^2 , *J. Geom. Anal.*, **21** (2011), 495–512.
- [Ke71] **N. Kerzman**—Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains, *Comm. Pure Appl. Math.*, **24** (1971), 301–379.
- [Kh10] **T. V. Khanh**—A general method of weights in the $\bar{\partial}$ -Neumann problem. *Ph.D. thesis. arXiv:1001.5093v1* (2010).
- [Ko02] **J. J. Kohn**—Superlogarithmic estimates on pseudoconvex domains and CR manifolds. *Ann. of Math.*, **156** (2002), 213248.
- [Kr76] **S. Krantz**—Optimal Lipschitz and L^p estimates for the equation $\bar{\partial}u = f$ on strongly pseudoconvex domains, *Math. Ann.*, **219** (1976), 233–260.
- [KZ10] **T. V. Khanh, G. Zampieri**—Regularity of the $\bar{\partial}$ -Neumann problem at point of infinite type. *J. Funct. Anal.*, **259** (2010), 2760–2775.
- [H02] **T. Hefer**—Hölder and L^p estimates for $\bar{\partial}$ on convex domains of finite type depending on Catlin’s multitype, *Math. Z.*, **242** (2002), 367–398.
- [Mc91] **J. D. McNeal**—On sharp Hölder estimates for the solutions of the $\bar{\partial}$ -equations. Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989), 277–285, Proc. Sympos. Pure Math., 52, Part 3, Amer. Math. Soc., Providence, RI, 1991.
- [He70] **G. M. Henkin**—Integral representation of functions in strictly pseudoconvex domains and applications to the $\bar{\partial}$ -problem, *Mat. Sb.*, **124**(1970), 300–308.
- [Ra86] **R. M. Range**—Holomorphic Functions and Integral Representations in Several Complex Variables, Springer-Verlag, Berlin/New York, 1986.
- [Ra90] **R.M. Range**—Integral kernels and Hölder estimates for $\bar{\partial}$ on pseudoconvex domains of finite type in \mathbb{C}^2 , *Math. Ann.*, **288** (1990), 63–74.